ON SOME PROPERTIES OF SUPERSONIC CONICAL GAS FLOW

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The linear theory of supersonic gas flow does not give sufficient accuracy for the determination of the aerodynamic characteristics of many bodies encountered in practice. For this reason more accurate solutions of the equations for the perturbation velocity potential Φ have been found by various authors (Van Dyke [1], Broderick [2], Moore [3], etc.) for bodies lying entirely inside the Mach cone produced in an undisturbed homogeneous gas stream. But in all these works no general analysis is given of the effect upon the flow field of replacing the conditions at the shock wave by the condition $\Phi = 0$ at the Mach cone of the undisturbed stream.

An analysis is given below of the effect upon the conical flow field in higher approximations of the shock wave produced by a conical body lying entirely inside the Mach cone produced in an undisturbed uniform supersonic gas stream. By the methods of the theory of the "boundary layer", series in a small parameter are found that represent the conical potential F in the vicinity of a weak shock wave and in the vicinity of the Mach cone (when an expansion of the undisturbed stream occurs). Then these series are "matched" with the series in a small parameter (relative thickness, angle of attack, etc.) representing F in the "inner" part of the flow, which makes it possible to obtain the flow field in the entire region between the body and the shock wave. It is known [4] that the series in a small parameter representing the exact solution in the "inner" part of the flow diverges in the vicinity of the shock wave (or Mach cone). Analysis of the solution in the "boundary layer" shows that with formal analytical continuation of each term of this series to the undisturbed Mach cone, the conical perturbation velocity potential should vanish there up to terms of $O(\epsilon^8)$ for slender bodies (such as a circular cone) and to $O(\delta^4)$ for thin bodies (such as a triangular plate), where ϵ is the thickness ratio and δ the angle of attack. (Higher-order terms in the series are infinite on the Mach cone.) Vorticity appears in terms $O(\epsilon^{12})$ and $O(\delta^{6})$ respectively for slender and thin bodies. The position of the shock wave produced by the body is also determined. The result agrees with the result of Lighthill [4].

1. Let a uniform supersonic stream of gas flow past a conical body with speed W_1, Mach number M_1, and speed of sound a_1. We assume that the body lies completely within the Mach cone of the undisturbed stream with vertex at the tip of the body. A conical flow is produced around the body, in which the velocity, entropy and pressure are constant on rays passing through the vertex of the body. The conical flow is separated from the uniform stream by a shock wave, or partly by a shock wave and partly by a Mach cone in the region where the undisturbed stream expands. We place the origin of a Cartesian coordinate system at the vertex of the conical body, with the z-axis in the direction of the undisturbed stream. As independent variables in the conical flow we take $\xi = x/z$ and $\eta = y/z$; henceforth we shall use the polar coordinates $\xi = r \cos\theta$ and $\eta = r \sin\theta$ in the $\xi\eta$-plane.

The velocity potential $\phi(x, y, z)$ is represented for conical flow by $\phi = zF(r, \theta)$, where F satisfies the equation [5]

$$\{a^2(1+r^2) - [rF - (1+r^2)F_r]^2\}F_{rr} + 2\Big[F - r\Big(\frac{1}{r}+r\Big)\Big]F_\theta\Big(\frac{1}{r}F_{r\theta} - \frac{1}{r^2}F_\theta\Big) + \Big(a^2 - \frac{1}{r^2}F_\theta^2\Big)\Big(\frac{1}{r^2}F_{\theta\theta} + \frac{1}{r}F_r\Big) = 0 \qquad (1.1)$$

$$a^2 = a_1{}^2 - \frac{\gamma-1}{2}\Big[F_r{}^2 + \frac{1}{r^2}F_\theta{}^2 + (F - rF_r)^2 - W_1{}^2\Big]$$

Here a is the speed of sound and γ the adiabatic index. (The vorticity of the flow shows no effect upon the following results.)

2. We find an expansion of the conical potential F as a series in a small parameter in the vicinity of a weak shock wave, and in the vicinity of the Mach cone when the undisturbed flow is expanded. At the Mach cone

$$r = r_1 = \frac{1}{m_1}, \qquad m_1 = \sqrt{M_1{}^2 - 1}$$

Here the perturbation velocity vanishes; that is, at $r = r_1$, $F = W_1$ and $F_r = 0$. We write the equation of the shock wave in the form $r = r_s(\theta) = r_1 + \lambda\phi(\theta, \lambda)$, where λ is a small parameter and $\phi(\theta, 0) = \phi_0(\theta) < \infty$.

For an irrotational flow the condition of continuity of the tangential component of velocity at the shock and Prandtl's condition on the normal component may be written in the form

$$r = r_s(\theta), \qquad F = W_1, \qquad F_r = -\frac{2W_1}{(\gamma+1)M_1^2}\,\frac{r_s{}^2(1 - r^2{}_s m_1{}^2) + r_s{}'^2}{r_s[r^2{}_s(1 + r_s{}^2) + r_s{}'^2]} \qquad (2.1)$$

Here the prime indicates differentiation with respect to θ.

For $r = r_s(\theta)$ one can obtain from (1.1) and (2.1) respectively

$$F_r = \frac{4W_1}{\gamma+1}\frac{m_1^4}{M_1^4}\lambda\varphi + O(\lambda^2), \qquad F_{rr} = -\frac{2W_1}{\gamma+1}\frac{m_1^4}{M_1^4} + O(\lambda)$$

This shows that near the shock wave there appears a "boundary layer" such that if the thickness and angle of attack of the body producing the shock tend to zero, the gradient of the modulus of the velocity vector in the neighborhood of the shock wave remains finite ($F_{rr} \neq 0$), whereas it tends to zero in the rest of the flow. The appearance of the "boundary layer" is a result of the fact that the coefficient of F_{rr} in Equation (1.1) is of the small order $0(\lambda)$ in the vicinity of the shock wave; that is, we have here an equation with a small parameter in the highest derivative (in the direction of r, almost normal to the boundary, the shock wave). By analogy with the Prandtl boundary layer in a viscous fluid, it is natural to call this phenomenon a "boundary layer," and to use the methods of boundary-layer theory to find the flow in the neighborhood of the shock wave. A "boundary layer" also exists near the Mach cone, since for an arbitrarily small perturbation in the vicinity of the Mach cone, F is represented by the expansion [5]

$$F = W_1 + \beta_1 (r_1 - r)^2 + \gamma_1 (r_1 - r)^3 \ln (r_1 - r) + C(\theta)(r_1 - r)^3 + \ldots \qquad (2.2)$$

where

$$\beta_1 = \frac{W_1}{2(\gamma+1)}\frac{m_1^4}{M_1^4}, \qquad \gamma_1 = \frac{W_1}{6(\gamma+1)^2}\frac{m_1^5}{M_1^6}[3m_1^2 - (\gamma+1)(1-m_1^2)]$$

$C(\theta)$ is an arbitrary function; $F_{rr} = 2\beta_1 \neq 0$ at $r = r_1$.

We shall seek F in the vicinity of the shock wave in the form of a series

$$F = W_1 + \lambda^2 F_1(\theta, t, \lambda) + \lambda^2 F_2(\theta, t, \lambda) + \ldots \qquad \left(t = \frac{r - r_1}{\lambda\varphi(\theta, \lambda)}\right) \qquad (2.3)$$

The value $t = 1$ corresponds to the shock wave. Substituting (2.3) into (1.1) we obtain equations for F_1, F_2, ... If a function $y(t)$ is introduced according to

$$F_1 = \varphi^2(\theta, \lambda)\frac{W_1}{\gamma+1}\frac{m_1^4}{M_1^4} y(t)$$

then the function $y(t)$ satisfies the equation

$$y''(2t - y') = y' \qquad (2.4)$$

The equations for F_k are transformed into

$$\frac{\partial^2 F_k}{\partial t^2}(y' - 2t) + \frac{\partial F_k}{\partial t}(y'' + 1) = P_k \qquad (2.5)$$

Here the P_k depend upon $F_1, \ldots, F_{k-1}$. The conditions (2.1) on the shock wave at $t = 1$ give

$$F_1 = 0, \qquad \frac{\partial F_1}{\partial t} = \frac{4W_1}{\gamma + 1}\frac{m_1^4}{M_1^4}\varphi^2(\theta, \lambda) \qquad (2.6)$$

$$F_2 = 0, \qquad \frac{\partial F_2}{\partial t} = -\frac{2W_1}{\gamma+1}\frac{m_1^5}{M_1^4}\varphi(\theta, \lambda)\left[\varphi'^2(\theta, \lambda) + \frac{M_1^2 + 4}{M_1^2}\varphi^2(\theta, \lambda)\right] \qquad \text{etc.}$$

From (2.4) and (2.6) we obtain

$$y(t) = \tfrac{8}{3}\left[t - \tfrac{8}{9}\left(1 - \tfrac{3}{4}t\right)^{3/2} - \tfrac{8}{9}\right] \qquad (2.7)$$

The functions P_k are sums of products of functions of t and of θ; therefore all the equations (2.5) can be integrated in closed form, since their integration reduces to the integration of simple ordinary differential equations.

In particular, for F_2 we obtain

$$F_2 = \frac{W_1 m_1^5}{(\gamma+1)M_1^4}\{\varphi^3(\theta, \lambda)x_1(t) + \varphi(\theta, \lambda)\varphi'^2(\theta, \lambda)x_2(t) + \varphi^2(\theta, \lambda)\varphi''(\theta, \lambda)x_3(t)\} \qquad (2.8)$$

where x_1, x_2, x_3 satisfy the equations

$$x_1''(y' - 2t) + x_1'(y'' + 1) = y''\left[t^2 - ty' + \frac{m_1^2}{2(\gamma+1)M_1^2}y'^2 - \frac{2}{M_1^2}ty' + \frac{2 + (\gamma - 1)M_1^2}{(\gamma+1)M_1^2}y\right] + ty' - \frac{\gamma - 1}{\gamma + 1}\frac{m_1^2}{M_1^2}y \qquad (2.9)$$

$$x_2''(y' - 2t) + x_2'(y'' + 1) = -t^2y'' + 2ty' - 2y$$

$$x_3''(y' - 2t) + x_3'(y'' + 1) = ty' - 2y$$

with boundary conditions

$$x_1(t) = x_2(1) = x_3(1) = 0, \quad x_2'(1) = -2\frac{M_1^2 + 4}{M_1^2}, \quad x_2'(1) = -2, \quad x_3'(1) = 0$$

From (2.9) it is easy to obtain the asymptotic representation of F_2 for large t

$$F_2 = -\frac{W_1 m_1^5}{(\gamma+1)M_1^4}\frac{2\sqrt{3}}{45}(9\varphi^3 + \varphi\varphi'^2 - 2\varphi^2\varphi'')(-t)^{5/2} + \ldots \qquad (2.10)$$

and establish the general structure of F_2 for large t

$$F_2 = b_1(-t)^{5/2} + b_2(-t)^2 + b_3(-t)^{3/2}\ln(-t) + \ldots \qquad (2.11)$$

Here the b_k are certain functions of θ and λ (whose explicit representation is not required). The term in $\ln(-t)$ appears on account of $x_1(t)$. We consider the "boundary layer" in the vicinity of the Mach cone $r = r_1$. As before, we shall seek F in the form (2.3), but here $\phi(\theta, \lambda)$ is some undetermined function and λ a parameter characterizing the order of the velocity upon leaving the "boundary layer." Equations (2.4), (2.5) and (2.9) retain their form; only the boundary conditions change. At $t = 0$, corresponding to the Mach cone, $F_k = \partial F_k/\partial t = 0$. For $y(t)$ one obtains the equation

$$y(t) = \frac{1}{c}\left\{t + \frac{1}{3c}\left[(1-2ct)^{3/2} - 1\right]\right\} \qquad (2.12)$$

where c is an arbitrary constant. The boundary conditions (2.9) have the form

$$x_1(0) = x_2(0) = x_3(0) = x_1'(0) = x_2'(0) = x_3'(0) = 0$$

For small t we have $y(t) = 1/2\ t^2 + \ldots$; the solution of the homogeneous equation $x_0''(y' - 2t) + x_0'(y'' + 1) = 0$ has the form

$$x_0 = \frac{t^3}{3} + \ldots, \qquad x_1 = \frac{(\gamma+4)(m_1^2-1)+3}{6(\gamma+1)M_1^2}\, t^2 \ln(-t) + \ldots, \quad x_2, \quad x_3 \sim O(t^3)$$

Hence it follows that upon transforming to $r_1 - r$ and θ the expansion (2.3) agrees with (2.2), which shows the correctness of (2.3). For large t the solution behaves analogously to the case of the shock wave. In particular, Equation (2.11) is preserved.

3. If, now, in (2.3) we transform to $r_1 - r$ and θ for large t, the terms $\lambda^k F_k(\theta, t, \lambda)$ are all of order $O(\lambda^{1/2})$, and from (2.3) we obtain

$$F = W_1 - \lambda^{1/2}\varphi^{1/2}\frac{W_1}{\gamma+1}\frac{m_1^4}{M_1^4}\frac{8\sqrt{3}}{9}\left\{(r_1-r)^{3/2} + \frac{9}{20}m_1\left[1 + \frac{\varphi'^2 - 2\varphi\varphi''}{9\varphi^2}\right](r_1-r)^{5/2} + \ldots\right\} + O(\lambda) \qquad (3.1)$$

$$w = W_1 - \lambda^{1/2}\varphi^{1/2}\frac{W_1}{\gamma+1}\frac{m_1^{5/2}}{M_1^4}\frac{4\sqrt{3}}{9} \times \left\{(r_1-r)^{1/2} + \frac{5\varphi^2 + \varphi'^2 - 2\varphi\varphi''}{12\varphi^2}(r_1-r)^{3/2} + \ldots\right\} + O(\lambda) \qquad (3.2)$$

where w is the velocity component along the Oz-axis in the case of the shock wave. In the case of the Mach cone we obtain

$$F = W_1 + \lambda^{1/2}\psi^{1/2}\frac{W_1}{\gamma+1}\frac{m_1^4}{M_1^4}\frac{8\sqrt{3}}{9}\times \tag{3.3}$$
$$\times\left\{(r_1-r)^{3/2}+\frac{9}{20}m_1\left[1+\frac{\psi'^2-2\psi\psi''}{9\psi^2}\right](r_1-r)^{5/2}+\ldots\right\}+O(\lambda)$$

$$w = W_1 + \lambda^{1/2}\psi^{1/2}\frac{W_1}{\gamma+1}\frac{m_1^{3/2}}{M_1^4}\frac{4\sqrt{3}}{9}\times \tag{3.4}$$
$$\times\left\{(r_1-r)^{1/2}+\frac{5\psi^2+\psi'^2-2\psi\psi''}{12\psi^2}(r_1-r)^{3/2}+\ldots\right\}+O(\lambda)$$
$$\left(\psi^{1/2}(\theta,\lambda)=\frac{3}{2\sqrt{2c}}\varphi^{1/2}(\theta,\lambda)\right)$$

Expressions (3.1), (3.3); (3.2), (3.4) differ only in the sign of $\lambda^{1/2}$ and that fact that ϕ is replaced by ψ. From these expressions it follows that if they are valid for small but finite values of $(r_1 - r)^{1/2}$ and represent the linear solution that holds in the "inner" part of the flow, then their formal analytic continuation to $r = r_1$ gives zero values for the potential and perturbation velocity at $r = r_1$. In linear theory it is shown that $w - W_1$ is a harmonic function of the variables

$$\alpha = \frac{1-\sqrt{1-m_1^2r^2}}{m_1 r}, \quad \theta$$

But any harmonic function that vanishes at $\alpha = 1$ $(r = r_1)$ can be represented in the form

$$w - W_1 = c_0 \ln\alpha + \sum_{n=1}^{\infty}\left(\frac{1}{\alpha^n}-\alpha^n\right)(c_{n1}\cos n\theta + c_{n2}\sin n\theta)$$

where c_0, c_{n1}, c_{n2} are constants. In the vicinity of the Mach cone it reduces to a series in powers of $(r_1 - r)^{1/2}$

$$w = W_1 + \left[-c_0\sqrt{2}+\sum 2\sqrt{2n}\,(c_{n1}\cos n\theta + c_{n2}\sin n\theta)\right]\times$$
$$\times\left\{(r_1-r)^{1/2}+\frac{-5c_0+\sum 2n(5+4n^2)(c_{n1}\cos n\theta+c_{n2}\sin n\theta)}{12\left[-c_0+\sum 2n(c_{n1}\cos n\theta+c_{n2}\sin n\theta)\right]}(r_1-r)^{3/2}+\ldots\right\}$$
$$(n = 1, 2, \ldots, \infty) \tag{3.5}$$

If the expansions (3.2), (3.4) are valid for small but finite values of $(r_1 - r)^{1/2}$, then (3.5) should agree with (3.4) if the coefficient of $(r_1 - r)^{1/2}$ is positive and with (3.2) if that coefficient is negative; that is, the following equalities should hold:

$$-\lambda^{1/2}\varphi^{1/2}\frac{W_1}{\gamma+1}\frac{m_1^{3/2}}{M_1^4}\frac{4\sqrt{3}}{9} = -c_0\sqrt{2}+\sum 2\sqrt{2}\,n\,(c_{n1}\cos n\theta + c_{n2}\sin n\theta) \tag{3.6}$$

$$\frac{5\varphi^2+\varphi'^2-2\varphi\varphi''}{\varphi^2} = \frac{-5c_0+\sum 2n(5+4n^2)(c_{n1}\cos n\theta+c_{n2}\sin n\theta)}{-c_0+\sum 2n(c_{n1}\cos n\theta+c_{n2}\sin n\theta)} \quad (n=1,\ldots,\infty) \tag{3.7}$$

Analogously for ψ. Putting the left side of (3.7) in the form $5 - (\phi'/\phi)^2 - 2(\phi'/\phi)'$, it is easy to show that (3.7) is a consequence of (3.6) (and the same for ψ). (This fact was also verified by the author for special cases: the triangular plate at angle of attack and the circular cone.)

The above considerations provide a basis for assuming that (3.1) and (3.3) represent F for small but finite $(r_1 - r)^{1/2}$. In particular, it follows that the usual linear theory gives the correct result for F and the perturbation velocity inside the Mach cone. We now write (2.3) for large t, using (2.7), (2.12), (2.11), in the form

$$\begin{aligned} F &= W_1 + \lambda^2 F_1 + \lambda^3 F_2 + \ldots \\ &= W_1 + \lambda^2 [a_1 (-t)^{3/2} + a_2 (-t) + a_3 (-t)^{1/2} + a_4 + a_5 (-t)^{-1/2} + \ldots] + \\ &\quad + \lambda^3 [b_1 (-t)^{5/2} + b_2 (-t^2) + b_3 (-t)^{3/2} \ln (-t) + \ldots] + \ldots \qquad (3.8) \end{aligned}$$

Here the $a_k = a_k(\theta, \lambda)$ are the coefficients of the expansion of F_1 in decreasing powers of t. Upon transforming to $r_1 - r$ and θ for large t, we obtain from (3.8)

$$\begin{aligned} F = W_1 &+ \lambda^{1/2} \left[\frac{a_1}{\varphi^{3/2}} (r_1 - r)^{3/2} + \frac{b_1}{\varphi^{5/2}} (r_1 - r)^{5/2} + \ldots \right] + \\ &+ \lambda \left[\frac{a_2}{\varphi} (r_1 - r) + \ldots \right] + \lambda^{3/2} \ln \lambda \left[- \frac{b_3}{\varphi^{3/2}} (r_1 - r)^{3/2} + \ldots \right] + \qquad (3.9) \\ &+ \lambda^{3/2} \left[\frac{a_3}{\varphi^{1/2}} (r_1 - r)^{1/2} + \ldots \right] + \lambda^2 [a_4 + \ldots] + \lambda^{5/2} [a_5 \varphi^{1/2} (r_1 - r)^{-1/2} + \ldots] + \ldots \end{aligned}$$

Here the coefficients $a_k(\theta, \lambda)$, $b_k(\theta, \lambda)$ can in turn be represented as series in λ and $\ln \lambda$. From the expansion (3.9) that represents F for small but finite $(r_1 - r)^{1/2}$ it follows that with formal analytic continuation of each term of the expansion to the Mach cone ($r = r_1$) the perturbation velocity potential ($F - W_1$) vanishes at $r = r_1$ up to quantities of $O(\lambda^2)$. Terms of higher order than λ^2 become infinite as $r \to r_1$.

This means that the series in powers of $\lambda^{1/2}$ and $\ln \lambda$, representing $F - W_1$ in the "inner" part of the flow should, upon formal analytic continuation to $r = r_1$, vanish at $r = r_1$ with the inclusion of terms of magnitude $O(\lambda^2)$. Terms of the expansion of order higher than λ^2 become infinite (to a definite order) as $r \to r_1$.

4. We assume that the solution of the linear problem is known under the conditions that the potential and the perturbation velocity are equal to zero at $r = r_1$. In the vicinity of $r = r_1$ every such solution can be represented in the form

$$F = W_1\left[1 - \frac{2}{3} m_1^{3/2} A(\theta)(r_1 - r)^{3/2} + \ldots\right] \tag{4.1}$$

(The factor W_1 $2/3$ $m_1^{3/2}$ is isolated for convenient comparison with the results of Lighthill [4].) On the other hand, from (3.1) and (3.3) we have

$$F = W_1 - [\lambda\varphi_0(\theta)]^{1/2} \frac{W_1}{\gamma+1} \frac{m_1^4}{M_1^4} \frac{8\sqrt{3}}{9} (r_1 - r)^{3/2} + \ldots \tag{4.2}$$

$$F = W_1 + [\lambda\psi_0(\theta)]^{1/2} \frac{W_1}{\gamma+1} \frac{m_1^4}{M_1^4} \frac{8\sqrt{3}}{9} (r_1 - r)^{3/2} + \ldots \tag{4.3}$$

For those θ for which $A(\theta) > 0$, that is, the flow is compressed in the vicinity of $r = r_1$, we obtain by equating coefficients of $(r_1 - r)^{3/2}$ in (4.2) and (4.1)

$$\lambda\varphi_0(\theta) = \frac{3}{16}(\gamma+1)^2 \frac{M_1^8}{m_1^5} A^2(\theta) \tag{4.4}$$

For $r_s(\theta)$ one obtains the equation

$$m_1 r_s(\theta) = m_1[r_1 + \lambda\varphi_0(\theta) + \ldots] = 1 + \frac{3}{16}(\gamma+1)^2 M_1^8 m_1^{-4} A^2(\theta) + \ldots \tag{4.5}$$

which agrees with the formula of Lighthill [4].

For those θ for which $A(\theta) < 0$, that is, the flow is expanded in the vicinity of $r = r_1$, we obtain by equating coefficients of $(r_1 - r)^{3/2}$ in (4.3) and (4.1), analogous to (4.4), the quantity $\lambda\psi_0(\theta)$, which leads to the solution in the "boundary layer" and remains undetermined. This function characterizes the velocity of expansion of the stream in the vicinity of the Mach cone. The agreement obtained with the result of Lighthill is explained by the fact that Lighthill's method gives the correct leading term in the vicinity of the shock wave. If we transform Lighthill's solution in the vicinity of the shock wave to the variables t and θ of the "boundary layer", we obtain

$$F = W_1 + W_1 \frac{3}{32}(\gamma+1)^3 M_1^{12} m_1^{-6} A^4(\theta)\left[t - \frac{8}{9}\left(1 - \frac{3}{4}t\right)^{3/4} + \frac{4}{3}\right] + \ldots \tag{4.6}$$

From the relation (2.3) one obtains

$$F = W_1 + \lambda^2 F_1 + \ldots$$
$$= W_1 + \frac{8}{3}\frac{W_1}{\gamma+1}\frac{m_1^4}{M_1^4}\lambda^2\varphi^2\left[t - \frac{8}{9}\left(1 - \frac{3}{4}t\right)^{3/2} - \frac{8}{9}\right] + \ldots \tag{4.7}$$

Although the perturbation velocity potential $F - W_1$ does not vanish on the shock wave, at $t = 1$, according to Equation (4.6), the derivatives

$\partial F/\partial t$ that determine the position of the shock are identical according to (4.6) and (4.7). These results confirm the possibility of using Lighthill's method to determine the shock-wave position.

5. The author has investigated the effect of vorticity in the stream upon the velocity components in the "boundary layer" and those leaving it, starting from the full system of equations for the three components of velocity and the entropy [6], and has found that the influence of entropy upon the velocity components in the "boundary layer" appears in terms of order $O(\lambda^4)$, and $O(\lambda^3)$ for those leaving it, where $\lambda = \epsilon^4$ for a slender body, ϵ being the thickness ratio, and $\lambda = \delta^2$ for a thin body, δ being the angle of attack, etc. The entropy is $O(\lambda^3)$ everywhere in the stream.

6. It is well known that for flow past a slender body the order of the perturbation velocity is different near the surface of the body and in the "middle" part of the flow. For example, for a circular cone the perturbation velocity potential $F - W_1$ has in the "middle" part of the flow the order $O(\mu^2)$, and near the body $O(\mu^2 \ln \mu)$, where $\mu = \tan \epsilon$, ϵ being the semi-vertex angle. Strictly speaking, there is a "boundary layer" near the surface of a slender body. The author has investigated this "boundary layer" for the circular cone at zero angle of attack.

The potential F was taken in the form

$$F = W_1 + F_{2,1}(\tau)\mu^2 \ln \mu + F_2(\tau)\mu^2 + F_{4,1}(\tau)\mu^4 \ln \mu + \ldots \qquad (\tau = r\mu^{-1}) \qquad (6.1)$$

Calculation showed that the functions $F_{k,l}(r)$, $F_k(r)$ are sums of products of positive powers of $\ln r$ and powers of r. These functions make it possible to separate r and μ for all r, and it is consequently possible to represent F as a single expansion in powers of μ and $\ln \mu$ for the solution near the surface of the body and in the "middle" part of the flow; that is, this "boundary layer" is non-essential.

The situation is different for the "boundary layer" near the shock wave (or Mach cone). A term $(1 - 3/4\ t)^{3/2}$ appears in the expression for $F_1(t, \theta, \lambda)$, which does not admit of a single expansion, since for $|t| < 4/3$ this expression reduces to a series in powers of t, but for $|t| > 4/3$ to one in powers of t^{-1} (the shock wave corresponding to $t = 1$). For this reason the "boundary layer" near the shock wave has an essential character.

7. The practical conclusions from the results formulated above are that to obtain higher-order approximations in the problem of flow past a conical body it is necessary to seek the perturbation velocity potential as a series in a small parameter, and require that each term

of this expansion vanish at the Mach cone of the undisturbed stream up to terms of $O(\epsilon^8)$ for a slender body and $O(\delta^4)$ for a thin body (where ϵ is the thickness ratio, and δ the angle of attack or other appropriate parameter). There is reason to suppose that this result holds also for the general case of flow past a body situated within the Mach cone of an undisturbed uniform supersonic stream of gas.

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